

On derivative interactions for a spin-2 field at cubic order

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ABSTRACT: Lorentz invariant derivative interactions for a single spin-2 field are investigated, up to the cubic order. We start from the most general Lorentz invariant derivative terms, which are polynomials in the spin-2 field as well as its first spacetime derivatives. Using a perturbative ADM analysis, we determined the parameters such that the corresponding Hamiltonian possesses a Lagrange multiplier, which would signify there are at most 5 degrees of freedom are propagating. The resulting derivative terms are linear combinations of terms coming from the expansion of Einstein-Hilbert Lagrangian around a Minkowski background, as well as the cubic “pseudo-linear derivative term” identified in [20]. We also derived the compatible potential terms, which are linear combinations of the expansions of the first two dRGT mass terms in unitary gauge.

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1 Introduction

Remarkable progresses have been made in the study of ghost-free nonlinear massive gravity very recently. In 4 dimensions, a massive spin-2 particle should carry 5 degrees of freedom and any consistent theory for massive gravity must satisfy this requirement. The quadratic order mass term was firstly constructed by Fierz and Pauli (FP) [1], which together with the linearized Einstein-Hilbert term describes a free massive spin-2 particle propagating on a Minkowski background. In the FP theory, however, the helicity-0 mode couples to the trace of the matter energy-momentum tensor with the same strength as the helicity-2 modes, which prevents the theory from recovering linearized General Relativity (GR) in the massless limit, known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity [2, 3]. It was argued that this discontinuity could be avoided through the Vainshtein mechanism [4], where the nonlinear interactions become important in the massless limit. However, when accompanied with the fully nonlinear Einstein-Hilbert term, the Fierz-Pauli mass term as well as its “naive” higher order generalizations will propagate a sixth degree of freedom at nonlinear orders, known as the Boulware-Deser (BD) ghost [5, 6], which implies the classical instability of the theory.

Counting degrees of freedom can be systematically performed in the Hamiltonian analysis. For GR, the existence of the Hamiltonian and momentum constraints is transparent in the Arnowitt-Deser-Misner (ADM) formalism, where the lapse N and shift N_i enter the Hamiltonian linearly. For massive

gravity, however, things become subtle. First, for a general mass term, which could be *a priori* an arbitrary function of the metric, the lapse and shift appear nonlinearly and thus no longer act directly as Lagrange multipliers [7]. On the other hand, there is no reason that the constraint must only be generated by the lapse itself. Instead, it may be generated by the set of lapse and shift together, i.e. some combination of the lapse and the shift. This possibility was realized only very recently. Using the nonlinear Stückelberg method developed in [8], consistent generalizations of FP term were constructed up to the quintic order in [9], and resummed into a fully nonlinear form by de Rham, Gabadadze and Tolley (dRGT) [10] (see [11–13] for recent reviews). The dRGT theory possesses a Hamiltonian constraint (as well as a secondary constraint) necessary to remove the BD ghost [14, 15] (see also [16] for the constraint analysis in a covariant manner).

The dRGT mass terms are determined under the assumption that the kinetic term for gravity is GR, which is the unique theory for a single massless spin-2 field. However, for a massive spin-2 field, diffeomorphism invariance is broken and thus one is allowed to consider other diffeomorphism non-invariant kinetic (derivative) terms. Conversely, if the kinetic term is different from GR, there is a possibility that the mass term compatible with this non-GR kinetic term is also different from the dRGT mass term. If this is true, we will have a more general class of massive gravity theories beyond the dRGT one. Moreover, in bi/multi-metric theories (e.g. [17]), metrics interact with each other only through potential terms, which take the dRGT form. The non-GR derivative terms, if exist, may also induce derivative interactions among different metrics, and thus extend our understandings of interacting spin-2 fields. This work is devoted to explore this possibility.

In [18] (see also [19]), non-GR cubic self-interaction terms with two spacetime derivatives was shown to exist using the helicity decomposition. Such a possibility was systematically investigated in [20] in arbitrary dimensions and with more than two derivatives, where a class of so-called “pseudo-linear” terms was identified. The “pseudo-linear” terms are nonlinear in $h_{\mu\nu}$, but are invariant under the linearized gauge symmetries. In d dimensions, there are $(d+1)$ mass terms satisfying this property:

$$\mathcal{L}_{0,n} \sim \delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} h_{\mu_1}^{\nu_1} \dots h_{\mu_n}^{\nu_n}, \quad n = 0, 1, \dots, d,$$

where the FP term is just $\mathcal{L}_{0,2}$, while $\mathcal{L}_{0,n}$ with $n \geq 3$ are the higher order analogue of FP term. The pseudo-linear terms with two derivatives are

$$\mathcal{L}_{2,n} \sim \delta_{\nu_1 \dots \nu_{n+1}}^{\mu_1 \dots \mu_{n+1}} \partial_{\mu_1} \partial^{\nu_1} h_{\mu_2}^{\nu_2} \dots h_{\mu_{n+1}}^{\nu_{n+1}}, \quad n = 1, \dots, d-1,$$

where the linearized Einstein-Hilbert term is nothing but $\mathcal{L}_{2,2}$, while the non-GR cubic term found in [18] just corresponds to $\mathcal{L}_{2,3}$ and is the unique pseudo-linear derivative term in 4 dimensions. In 4 dimensions, $\mathcal{L}_{0,2}$, $\mathcal{L}_{0,3}$ and $\mathcal{L}_{0,4}$ are the leading terms in the expansion of the corresponding dRGT mass terms around a Minkowski background. Conversely, the full dRGT mass terms can be viewed as the “nonlinear completion” of these pseudo-linear mass terms. It was thus conjectured that there is also a “nonlinear completion” of the pseudo-linear derivative term $\mathcal{L}_{2,3}$ [20].

Two types of nonlinear derivative terms were constructed in [21], $G^{\mu\nu} \mathcal{K}_{\mu\nu}$ and ${}^* R^{\mu\nu\rho\sigma} \mathcal{K}_{\mu\rho} \mathcal{K}_{\nu\sigma}$ ($\mathcal{K}_{\nu}^{\mu} \equiv \delta_{\nu}^{\mu} - \sqrt{\delta_{\nu}^{\mu} - H_{\nu}^{\mu}}$, where $H_{\mu\nu}$ is the covariantized metric perturbation $h_{\mu\nu}$ which reduces to $h_{\mu\nu}$ in unitary gauge, see e.g [9, 10] for details), which reproduce linearized Einstein-Hilbert term

$\mathcal{L}_{2,2}$ and the pseudo-linear derivative term $\mathcal{L}_{2,3}$ respectively when being expanded around Minkowski background. However, both terms were shown to suffer from a ghost at the energy scale Λ_3 in the decoupling limit. The same two nonlinear derivative terms were derived from a higher dimensional Gauss-Bonnet term in [22] using the “dimensional deconstruction” approach [23], which has successfully reproduced the dRGT mass terms from a higher dimensional Einstein-Hilbert term¹. Based on a perturbative analysis, a “no-go” theorem was further proposed in [22], which claims that the pseudo-linear derivative term $\mathcal{L}_{2,3}$ does not possess a ghost-free nonlinear completion, and thus there is no diffeomorphism-breaking but Lorentz invariant derivative terms in metric formulation of massive gravity.

In this note, we perform a perturbative ADM analysis to find possibly ghost-free derivative terms, up to the cubic order. We consider the most general Lorentz invariant derivative terms, which are polynomials in the spin-2 field $h_{\mu\nu}$ as well as its first spacetime derivatives. They describe a single massive spin-2 particle propagating on a Minkowski background. By tuning the parameters such that the Hamiltonian possesses a possible constraint which eliminates the ghost degree of freedom, we can determine the derivative terms as well as the compatible potential terms. In general, as is for the dRGT terms [14, 15], such a constraint is generated by a combination of $\{\delta N, N_i\}$, which are the perturbation of lapse function and the shift vector respectively. Our investigation can be viewed as a complementary but more systematic approach with respect to the analysis in [22], where first a nonlinear Stückelberg analysis was performed to determine the necessary choice of parameters, whose sufficiency was subsequently tested by a specific perturbative ADM analysis.

In Sec.2, we set up the necessary formalism for our perturbative ADM analysis for the Hamiltonian, and in Sec.2.2, as an example, we show how the dRGT mass terms can be determined perturbatively in our approach. In Sec.3, we determine the derivative terms as well as the compatible mass terms up to the cubic order. We summarize our results in Sec.4.

2 Perturbative Hamiltonian

In this work, we will deal with Lagrangian consisting of Lorentz invariant polynomials of a spin-2 field $h_{\mu\nu}$ and its first spacetime derivatives. Moreover, we restrict ourselves to the case with no more than two spacetime derivatives.

The derivation of Hamiltonian relies on the splitting of space and time. Thus it is convenient to use the ADM variables $\{N, N_i, \gamma_{ij}\}$, which are defined through

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = - (N^2 - \gamma^{ij} N_i N_j) dt dt + 2N_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad (2.1)$$

where N is the lapse, N_i is the shift, and γ^{ij} is the matrix inverse of the spatial metric γ_{ij} . The deviation from a Minkowski background is parametrized by

$$N \equiv 1 + \alpha, \quad N_i \equiv \beta_i, \quad (2.2)$$

¹Alternative terms consisting of $\mathcal{K}_{\mu\nu}$ and its covariant derivatives are studied in [24], which also have the ghost problem.

and $\gamma_{ij} \equiv \delta_{ij} + h_{ij}$, so that

$$\gamma^{ij} = \delta_{ij} - h_{ij} + h_{ik}h_{kj} + \mathcal{O}(h^3), \quad (2.3)$$

where throughout this work repeated lower spatial indices are summed using δ_{ij} . We use $\{\alpha, \beta_i, h_{ij}\}$ as the perturbative variables, for example, in terms of which h_{00} can be rewritten as

$$h_{00} = -\alpha(2 + \alpha) + (\delta_{ij} - h_{ij})\beta_i\beta_j + \dots, \quad (2.4)$$

where “ \dots ” denotes terms which are quartic (and higher) order in β_i and h_{ij} .

No matter massless or massive, it is the spatial component h_{ij} that carries the dynamical degrees of freedom in $h_{\mu\nu} \equiv \delta g_{\mu\nu}$. This means firstly we have to make sure that the Lagrangian can be expressed in the first order form, where only h_{ij} has time derivatives, while α and β_i have vanishing conjugate momenta and thus do not enter the phase space. As we will see, at least up to the cubic order, this simple requirement has already determined the consistent derivative terms².

After eliminating time derivatives on α and β_i , we will deal with a Lagrangian taking the following form

$$\mathcal{L}^{\text{ADM}} = \frac{1}{2}\mathcal{G}_{ij,kl}\dot{h}_{ij}\dot{h}_{kl} + \mathcal{F}_{ij}\dot{h}_{ij} + \mathcal{W}, \quad (2.5)$$

where a dot denotes time derivative, \mathcal{G} , \mathcal{F} and \mathcal{W} are functions of $\{\alpha, \beta_i, h_{ij}\}$ containing no time derivatives, schematically,

$$\mathcal{G}_{ij,kl} = \mathcal{G}_{ij,kl}(\alpha, \beta, h), \quad \mathcal{F}_{ij} = \mathcal{F}_{ij}(\alpha, \beta, h), \quad \mathcal{W} = \mathcal{W}(\alpha, \beta, h). \quad (2.6)$$

Note \mathcal{F}_{ij} is symmetric with respect to its indices, while $\mathcal{G}_{ij,kl}$ has the following symmetries

$$\mathcal{G}_{ij,kl} = \mathcal{G}_{ji,kl} = \mathcal{G}_{ij,lk} = \mathcal{G}_{kl,ij}. \quad (2.7)$$

The inverse of $\mathcal{G}_{ij,kl}$ is defined as

$$\mathcal{G}_{ij,kl}^{-1}\mathcal{G}_{kl,i'j'} = \mathbf{I}_{ij,i'j'}, \quad (2.8)$$

where $\mathbf{I}_{ij,i'j'}$ is the identity in the space of symmetric matrices:

$$\mathbf{I}_{ij,i'j'} \equiv \frac{1}{2}(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'}), \quad (2.9)$$

which has the same symmetries as in (2.7).

The conjugate momentum of h_{ij} is defined by

$$\pi_{ij} \equiv \frac{\partial \mathcal{L}^{\text{ADM}}}{\partial \dot{h}_{ij}} = \mathcal{G}_{ij,kl}\dot{h}_{kl} + \mathcal{F}_{ij}, \quad (2.10)$$

from which we solve

$$\dot{h}_{ij} = \mathcal{G}_{ij,kl}^{-1}(\pi_{kl} - \mathcal{F}_{kl}). \quad (2.11)$$

²While additional constraints, including the consistency conditions between the potential and derivative terms, must be included in order to determine the potential terms.

The Hamiltonian density is thus given by

$$\mathcal{H} \equiv \pi_{ij} \dot{h}_{ij} - \mathcal{L} = \frac{1}{2} (\pi_{ij} - \mathcal{F}_{ij}) \mathcal{G}_{ij,kl}^{-1} (\pi_{kl} - \mathcal{F}_{kl}) - \mathcal{W}. \quad (2.12)$$

In this work, we are dealing with perturbative theories, where $\mathcal{G}_{ij,kl}$ etc are polynomials of $\{\alpha, \beta_i, h_{ij}\}$ and can be expanded as

$$\mathcal{G}_{ij,kl} = \mathcal{G}_{ij,kl}^{(0)} + \mathcal{G}_{ij,kl}^{(1)} + \cdots, \quad (2.13)$$

$$\mathcal{F}_{ij} = \mathcal{F}_{ij}^{(1)} + \mathcal{F}_{ij}^{(2)} + \cdots, \quad (2.14)$$

$$\mathcal{W} = \mathcal{W}^{(2)} + \mathcal{W}^{(3)} + \cdots, \quad (2.15)$$

and

$$\mathcal{G}_{ij,kl}^{-1} = (\mathcal{G}^{-1})_{ij,kl}^{(0)} + (\mathcal{G}^{-1})_{ij,kl}^{(1)} + \cdots, \quad (2.16)$$

where $(\mathcal{G}^{-1})_{ij,kl}^{(0)} \equiv (\mathcal{G}^{(0)})_{ij,kl}^{-1}$, and superscript “ (n) ” denotes the order in perturbations. The corresponding Hamiltonian density is also expanded as

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots, \quad (2.17)$$

with

$$\mathcal{H}_2 = \frac{1}{2} (\pi_{ij} - \mathcal{F}_{ij}^{(1)}) (\mathcal{G}^{-1})_{ij,kl}^{(0)} (\pi_{kl} - \mathcal{F}_{kl}^{(1)}) - \mathcal{W}^{(2)}, \quad (2.18)$$

$$\mathcal{H}_3 = \frac{1}{2} (\pi_{ij} - \mathcal{F}_{ij}^{(1)}) (\mathcal{G}^{-1})_{ij,kl}^{(1)} (\pi_{kl} - \mathcal{F}_{kl}^{(1)}) - \mathcal{F}_{ij}^{(2)} (\mathcal{G}^{-1})_{ij,kl}^{(0)} (\pi_{kl} - \mathcal{F}_{kl}^{(1)}) - \mathcal{W}^{(3)}, \quad (2.19)$$

etc.

For general functions $\mathcal{G}_{ij,k,l}$ etc as in (2.6), neither α nor β_i enter \mathcal{H} linearly, or strictly speaking, the determinant of Hessian $\det \left(\frac{\partial^2 \mathcal{H}}{\partial n_a \partial n_b} \right)$ with $n_a \equiv \{\alpha, \beta_i\}$ does not vanish. There is no further constraint, and thus the system will propagate all six degrees of freedom of h_{ij} , which signifies the existence of the BD ghost. Fortunately and as we will see, it is possible to tune the parameters such that there exists a constraint generated by the set of four variables $\{\alpha, \beta_i\}$. This is also the essence in proving the vanishing of BD ghost for the dRGT mass terms [14, 15].

2.1 Linearized Einstein-Hilbert

As is well-known, the quadratic kinetic term is uniquely determined to be (see Appendix A):

$$\mathcal{L}_2^{\text{der}} = b_1 \left(\partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + 2 \partial_\mu h^{\mu\nu} \partial_\nu h - 2 \partial_\nu h^{\mu\nu} \partial^\lambda h_{\mu\lambda} - \partial_\mu h \partial^\mu h \right) \simeq -4b_1 \mathcal{L}_2^{\text{GR}}, \quad (2.20)$$

where $\mathcal{L}^{\text{GR}} = \sqrt{-g}R$ and a subscript “ $_2$ ” denotes the expansion at the quadratic order around the Minkowski background. Throughout this work, upper Lorentzian indices are raised by Minkowski metric $\eta^{\mu\nu}$. In (2.20) we keep the overall factor b_1 simply for later convenience. In terms of perturbative ADM variables $\{\alpha, \beta_i, h_{ij}\}$, at the quadratic order, we have

$$\mathcal{L}_2^{\text{der,ADM}} = \frac{1}{2} \mathcal{G}_{ij,kl}^{(0)} \dot{h}_{ij} \dot{h}_{kl} + \mathcal{F}_{ij}^{(1)} \dot{h}_{ij} + \mathcal{W}^{(2)}, \quad (2.21)$$

with

$$\mathcal{G}_{ij,kl}^{(0)} = b_1 (2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad (2.22)$$

$$\mathcal{F}_{ij}^{(1)} = 2b_1 (\partial_i\beta_j + \partial_j\beta_i - 2\partial_k\beta_k\delta_{ij}), \quad (2.23)$$

$$\begin{aligned} \mathcal{W}^{(2)} = b_1 \Big[& -2\partial_j h_{ij}\partial_k h_{ik} + \partial_j h_{ii}(-\partial_j h_{kk} + 2\partial_k h_{kj}) + \partial_k h_{ij}\partial_k h_{ij} \\ & + 2(\partial_i\beta_i\partial_j\beta_j - \partial_j\beta_i\partial_j\beta_i) + 4\partial_j\alpha(\partial_i h_{ij} - \partial_j h_{ii}) \Big]. \end{aligned} \quad (2.24)$$

At this point, we can easily determine

$$\left(\mathcal{G}^{(0)}\right)_{ij,kl}^{-1} = \frac{1}{4b_1} (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \quad (2.25)$$

Plugging the above into (2.18) and after some manipulations, the corresponding Hamiltonian density is given by

$$\begin{aligned} \mathcal{H}_2^{\text{der}} = & \frac{1}{8b_1} (\pi_{ii}\pi_{jj} - 2\pi_{ij}\pi_{ij}) + 2\pi_{ij}\partial_i\beta_j + 4b_1\alpha(\partial_i\partial_j h_{ij} - \partial^2 h_{ii}) \\ & - b_1 [-2\partial_j h_{ij}\partial_k h_{ik} + \partial_j h_{ii}(-\partial_j h_{kk} + 2\partial_k h_{kj}) + \partial_k h_{ij}\partial_k h_{ij}], \end{aligned} \quad (2.26)$$

up to total derivatives. (2.26) explicitly shows that, at the quadratic order, both α and β_i enters the Hamiltonian linearly and thus act as Lagrange multiplier.

2.2 Fierz-Pauli term and its cubic generalization

As a simple illustration of our strategy, now we show how the Fierz-Pauli term [1] and its cubic generalization are determined by imposing the appropriate constraint structure.

The most general Lorentz invariant quadratic and cubic potential terms are

$$\mathcal{L}_2^{\text{pot}} = b'_1 h_\mu^\mu h_\nu^\nu + b'_2 h_{\mu\nu} h^{\mu\nu}, \quad (2.27)$$

$$\mathcal{L}_3^{\text{pot}} = c'_1 h_\mu^\mu h_\nu^\nu h_\rho^\rho + c'_2 h_\rho^\rho h_\nu^\mu h_\mu^\nu + c'_3 h_\nu^\mu h_\rho^\nu h_\mu^\rho, \quad (2.28)$$

where b'_i 's and c'_i 's are constant parameters to be determined. In terms of ADM variables, at the quadratic order,

$$\mathcal{L}_2^{\text{pot,ADM}} = 4(b'_1 + b'_2)\alpha^2 + 4b'_1\alpha h_{ii} - 2b'_2\beta_i\beta_i + b'_1 h_{ii}h_{jj} + b'_2 h_{ij}h_{ij}, \quad (2.29)$$

while at the cubic order,

$$\begin{aligned} \mathcal{L}_3^{\text{pot,ADM}} = & 8(c'_1 + c'_2 + c'_3)\alpha^3 + 2(b'_1 + 6c'_1 + 2c'_2)\alpha^2 h_{ii} - 2(2c'_2 + 3c'_3)\alpha\beta_i\beta_i \\ & + 2\alpha(3c'_1 h_{ii}h_{jj} + c'_2 h_{ij}h_{ij}) - 3c'_3 h_{ij}\beta_i\beta_j - 2(b'_1 + c'_2)h_{ii}\beta_j\beta_j \\ & + c'_1 h_{ii}h_{jj}h_{kk} + c'_2 h_{ii}h_{jk}h_{jk} + c'_3 h_{ij}h_{jk}h_{ki}. \end{aligned} \quad (2.30)$$

Note $\mathcal{L}_2^{\text{pot}}$ also contributes to $\mathcal{L}_3^{\text{pot,ADM}}$.

In order to guarantee the possible existence of a constraint relevant to α , first the self-coupling terms of α must be vanishing, which implies, at the quadratic order,

$$b'_1 + b'_2 = 0, \quad (2.31)$$

and, at the cubic order,

$$c'_1 + c'_2 + c'_3 = 0. \quad (2.32)$$

At the quadratic order, (2.31) uniquely fixes the mass terms to the form of Fierz-Pauli, up to an overall factor b'_1 . While at the cubic order, there are two apparently problematic terms in (2.30), proportional to $\alpha^2 h_{ii}$ and $\alpha \beta_i \beta_i$ respectively, which seem to prevent α from being a Lagrange multiplier. These two terms, however, can be regrouped (together with the relevant terms in $\mathcal{L}_2^{\text{pot,ADM}}$) into

$$2b'_1 \left[\beta_i \beta_i - 2 \frac{2c'_2 + 3c'_3}{2b'_1} \alpha \beta_i \beta_i \right] \subset 2b'_1 \left(\beta_i - \frac{2c'_2 + 3c'_3}{2b'_1} \alpha \beta_i \right)^2, \quad (2.33)$$

and

$$4b'_1 \alpha h_{ii} + 2(b'_1 + 6c'_1 + 2c'_2) \alpha^2 h_{ii} = 4b'_1 \left(\alpha + \left(\frac{1}{2} + \frac{3c'_1 + c'_2}{b'_1} \right) \alpha^2 \right) h_{ii}. \quad (2.34)$$

Thus we can introduce new variables

$$\tilde{\beta}_i = \beta_i - \frac{2c'_2 + 3c'_3}{2b'_1} \alpha \beta_i, \quad (2.35)$$

$$\tilde{\alpha} = \alpha + \left(\frac{1}{2} + \frac{3c'_1 + c'_2}{b'_1} \right) \alpha^2. \quad (2.36)$$

With these redefined variables, $\tilde{\alpha}$ appears linearly, thus serves as a Lagrange multiplier. At this point, note since β_i (or $\tilde{\beta}_i$) has already appeared quadratically, $\tilde{\alpha} + \lambda \beta_i^2$ with arbitrary constant λ , is also a valid variable that acts as a Lagrange multiplier. This freedom will be used when matching the redefinition (2.36) for the potential terms with that for the derivative terms (see the discussion around (3.18)).

On the other hand, the potential terms must always be accompanied with the kinetic terms. If the kinetic term is taken to be GR, it is the lapse function N , or equivalently α , that plays the role as a Lagrange multiplier and generates the corresponding (Hamiltonian) constraint. The potential terms must be tuned to be compatible with this property. Thus, (2.36) implies another constraint among parameters

$$\frac{1}{2} + \frac{3c'_1 + c'_2}{b'_1} = 0, \quad (2.37)$$

which together with (2.32) yields

$$c'_2 = -\frac{1}{2}b'_1 - 3c'_1, \quad c'_3 = \frac{1}{2}b'_1 + 2c'_1,$$

with a single free parameter c'_1 left undetermined. Finally, up to the cubic order, the potential terms compatible with GR are

$$\begin{aligned} \mathcal{L}_2^{\text{pot}} + \mathcal{L}_3^{\text{pot}} = & b'_1 \left[(h^2 - h_{\mu\nu} h^{\mu\nu}) + \frac{1}{2} (h^3 - 4h h_{\mu\nu} h^{\mu\nu} + 3h_\nu^\mu h_\rho^\nu h_\mu^\rho) \right] \\ & + \left(c'_1 - \frac{1}{2}b'_1 \right) (h^3 - 3h h_{\mu\nu} h^{\mu\nu} + 2h_\nu^\mu h_\rho^\nu h_\mu^\rho), \end{aligned} \quad (2.38)$$

where $h \equiv h_\mu^\mu$. It is easy to check that, the first line in (2.38) is nothing but the expansions of the first dRGT mass term

$$\mathcal{L}^{\text{dRGT},1} \equiv \sqrt{-g} \left([\mathcal{K}]^2 - [\mathcal{K}^2] \right), \quad (2.39)$$

up to the third order in $h_{\mu\nu}$ around a flat background in unitary gauge, while the second line in (2.38) corresponds to the expansion of

$$\mathcal{L}^{\text{dRGT},2} \equiv \sqrt{-g} \left([\mathcal{K}]^3 - 3 [\mathcal{K}] [\mathcal{K}^2] + 2 [\mathcal{K}^3] \right). \quad (2.40)$$

Following this perturbative approach, one may in principle re-arrive at the nonlinear structure (or the nonlinear completion) for the dRGT mass terms found in [9, 10].

At this point, we emphasize again that the existence of the constraint (2.37) and thus (2.38) are determined under the assumption that, the kinetic term is given exactly by GR. In other words, the kinetic terms are given firstly, while the potential terms are added and tuned elaborately to be compatible with the constraint structure of the kinetic terms. There is a possibility, however, different potential terms (from the dRGT ones) may exist, but are compatible with different (non-GR) derivative terms.

3 Derivative interactions at the cubic order

We start from the most general Lorentz invariant cubic Lagrangian for $h_{\mu\nu}$ with two spacetime derivatives, which takes the form³

$$\begin{aligned} \mathcal{L}_3^{\text{der}} = & c_1 h^{\alpha\beta} \partial_\alpha h^{\lambda\mu} \partial_\beta h_{\lambda\mu} + c_2 h^{\alpha\beta} \partial_\alpha h_\lambda^\lambda \partial_\beta h_\mu^\mu + c_3 h^{\alpha\beta} \partial_\beta h_\alpha^\lambda \partial_\lambda h_\mu^\mu + c_4 h^{\alpha\beta} \partial_\lambda h_\mu^\mu \partial^\lambda h_{\alpha\beta} \\ & + c_5 h_\alpha^\alpha \partial_\lambda h_\mu^\mu \partial^\lambda h_\beta^\beta + c_6 h^{\alpha\beta} \partial_\lambda h_\alpha^\lambda \partial_\mu h_\beta^\mu + c_7 h^{\alpha\beta} \partial_\beta h_\alpha^\lambda \partial_\mu h_\lambda^\mu + c_8 h_\alpha^\alpha \partial_\beta h^{\beta\lambda} \partial_\mu h_\lambda^\mu \\ & + c_9 h^{\alpha\beta} \partial^\lambda h_{\alpha\beta} \partial_\mu h_\lambda^\mu + c_{10} h_\alpha^\alpha \partial^\lambda h_\beta^\beta \partial_\mu h_\lambda^\mu + c_{11} h^{\alpha\beta} \partial_\lambda h_{\beta\mu} \partial^\mu h_\alpha^\lambda + c_{12} h^{\alpha\beta} \partial_\mu h_{\beta\lambda} \partial^\mu h_\alpha^\lambda \\ & + c_{13} h_\alpha^\alpha \partial_\lambda h_{\beta\mu} \partial^\mu h^{\beta\lambda} + c_{14} h_\alpha^\alpha \partial_\mu h_{\beta\lambda} \partial^\mu h^{\beta\lambda}, \end{aligned} \quad (3.1)$$

where c_1, \dots, c_{14} are constant parameters to be determined.

Together with $\mathcal{L}_2^{\text{der}}$ given in (2.20), after a tedious calculation, the expansion of $\mathcal{L}_2^{\text{der}} + \mathcal{L}_3^{\text{der}}$ in terms of $\{\alpha, \beta_i, h_{ij}\}$ at the cubic order can be schematically classified into 10 groups

$$\begin{aligned} \mathcal{L}_3^{\text{der,ADM}} = & \mathcal{L}_{3,\alpha^3} + \mathcal{L}_{3,\alpha^2\beta} + \mathcal{L}_{3,\alpha\beta^2} + \mathcal{L}_{3,\beta^3} \\ & + \mathcal{L}_{3,\alpha^2h} + \mathcal{L}_{3,\alpha\beta h} + \mathcal{L}_{3,\beta^2h} + \mathcal{L}_{3,\alpha h^2} + \mathcal{L}_{3,\beta h^2} + \mathcal{L}_{3,h^3}, \end{aligned} \quad (3.2)$$

where the explicit expressions for each group of terms are given in Appendix B.1. Requiring that $\mathcal{L}_3^{\text{ADM}}$ can be put into the first order form with no time derivatives of α and β_i yields 12 constraints (see (B.11)-(B.22)) on the 14 parameters c_i 's, from which we solve:

$$\begin{aligned} c_2 &= -c_1, & c_3 &= 4c_1, & c_4 &= -2c_1, & c_6 &= -5c_1 + 2c_5, \\ c_7 &= -4c_1, & c_8 &= -3c_1 + 2c_5, & c_9 &= 2c_1, & c_{10} &= -2c_5, \\ c_{11} &= 3c_1 - 2c_5, & c_{12} &= 2c_1, & c_{13} &= 3c_1, & c_{14} &= -c_5, \end{aligned} \quad (3.3)$$

³Actually there are 16 independent contractions, among which 2 can be removed by linear combinations of other terms up to total derivatives. Thus here we choose the residual 14 terms the same as in [22] (with different orders).

with c_1 and c_5 left undetermined.

3.1 Cubic Hamiltonian \mathcal{H}_3

After plugging the solutions for parameters (3.3), the cubic Lagrangian (3.2) can be written as

$$\mathcal{L}_3^{\text{der,ADM}} = \frac{1}{2} \mathcal{G}_{ij,kl}^{(1)} \dot{h}_{ij} \dot{h}_{kl} + \mathcal{F}_{ij}^{(2)} \dot{h}_{ij} + \mathcal{W}^{(3)}, \quad (3.4)$$

where

$$\begin{aligned} \mathcal{G}_{ij,kl}^{(1)} = & (2(c_1 - c_5)\alpha - c_5 h_{mm}) (2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ & + c_1 (2h_{ij}\delta_{kl} + 2h_{kl}\delta_{ij} - h_{il}\delta_{jk} - h_{jl}\delta_{ik} - h_{ik}\delta_{jl} - h_{jk}\delta_{il}), \end{aligned} \quad (3.5)$$

and $\mathcal{F}_{ij}^{(2)}$ and $\mathcal{W}^{(3)}$ are given in B.2.

Finally, together with the quadratic Lagrangian (2.21), the Hamiltonian density at the cubic order can be written as

$$\mathcal{H}_3^{\text{der}} = \mathcal{H}_{3,\alpha\pi^2} + \mathcal{H}_{3,\alpha^2 h} + \mathcal{H}_{3,\beta\pi h} + \mathcal{H}_{3,\beta^2 h} + \mathcal{H}_{3,h\pi^2} - \mathcal{W}_{\alpha h^2}^{(3)} - \mathcal{W}_{h^3}^{(3)}, \quad (3.6)$$

where

$$\mathcal{H}_{3,\alpha\pi^2} = -\frac{1}{4b_1^2} (c_1 - c_5) \alpha (\pi_{ii}\pi_{jj} - 2\pi_{ij}\pi_{ij}), \quad (3.7)$$

$$\mathcal{H}_{3,\alpha^2 h} = 2(b_1 + 2(c_1 - c_5)) \alpha^2 (\partial_i \partial_j h_{ij} - \partial^2 h_{ii}), \quad (3.8)$$

$$\begin{aligned} \mathcal{H}_{3,\beta\pi h} = & \frac{1}{b_1} \beta_i \left[(c_1 - 2c_5) \left(\pi_{ij} (\partial_j h_{kk} - \partial_k h_{jk}) - \frac{1}{2} \pi_{jj} (\partial_i h_{kk} + \partial_k h_{ik}) \right) \right. \\ & \left. - c_1 \pi_{jk} \partial_i h_{jk} + (3c_1 - 2c_5) \pi_{jk} \partial_k h_{ij} \right], \end{aligned} \quad (3.9)$$

$$\begin{aligned} \mathcal{H}_{3,\beta^2 h} = & (c_1 - 2c_5) \left[h_{ii} (\partial_j \beta_j \partial_k \beta_k + \partial_j \beta_k \partial_k \beta_j + 2\beta_j \partial_k \partial_j \beta_k) \right. \\ & + 2h_{ij} (\beta_i \partial^2 \beta_j - \beta_i \partial_k \partial_j \beta_k - \beta_k \partial_i \partial_k \beta_j - \partial_j \beta_i \partial_k \beta_k - \partial_j \beta_k \partial_k \beta_i + \partial_k \beta_j \partial_k \beta_i) \left. \right] \\ & + (2b_1 + 3c_1 - 2c_5) \beta_k^2 (\partial^2 h_{ii} - \partial_i \partial_j h_{ij}), \end{aligned} \quad (3.10)$$

$$\mathcal{H}_{3,h\pi^2} = \frac{1}{8b_1^2} [c_5 h_{ii} (\pi_{jj}\pi_{kk} - 2\pi_{jk}\pi_{jk}) - 2c_1 h_{ij} (\pi_{ij}\pi_{kk} - 2\pi_{ik}\pi_{jk})], \quad (3.11)$$

and $\mathcal{W}_{\alpha h^2}^{(3)}$ and $\mathcal{W}_{h^3}^{(3)}$ are given in (B.27) and (B.29).

At the cubic order, due to the presence of $\mathcal{H}_{3,\alpha^2 h}$ and $\mathcal{H}_{3,\beta^2 h}$, both α and β_i appear quadratically in the Hamiltonian. However, α^2 and β_i^2 enter the Hamiltonian through a special combination (see (3.8) and the last line of (3.10))

$$\mathcal{H}_2 + \mathcal{H}_3 \supset 4b_1 \left[\alpha + \left(\frac{1}{2} + \frac{c_1 - c_5}{b_1} \right) \alpha^2 - \frac{1}{2} \left(1 + \frac{3c_1 - 2c_5}{2b_1} \right) \beta_k^2 \right] (\partial_i \partial_j h_{ij} - \partial^2 h_{ii}), \quad (3.12)$$

where we also include the relevant term from $\mathcal{H}_2^{\text{der}}$. Thus, we can always introduce a new variable⁴

$$\tilde{\alpha} \propto \alpha + \left(\frac{1}{2} + \frac{c_1 - c_5}{b_1} \right) \alpha^2 - \frac{1}{2} \left(1 + \frac{3c_1 - 2c_5}{2b_1} \right) \beta_k^2, \quad (3.13)$$

such that, up to cubic order, $\tilde{\alpha}$ appears linearly in the Hamiltonian. This also implies, there is no further constraint between c_1 and c_5 , in other words, simply requiring the vanishing of $\dot{\alpha}$ and $\dot{\beta}_i$ has already been sufficient to determine the derivative terms.

3.2 Derivative terms and the compatible potential terms

Up to the cubic order in $h_{\mu\nu}$, we are left with a set of (generally non-diffeomorphism invariant) derivative terms with two parameters, which can be written in the form⁵

$$\mathcal{L}_2^{\text{der}} + \mathcal{L}_3^{\text{der}} \simeq -4b_1 \left(\mathcal{L}_2^{\text{GR}} + 2 \frac{c_5 - c_1}{b_1} \mathcal{L}_3^{\text{GR}} + \frac{c_1 - 2c_5}{4b_1} \mathcal{L}^{\text{PL}} \right), \quad (3.14)$$

where \mathcal{L}^{GR} denotes $\sqrt{-g}R$, the subscripts denote the expansions in $h_{\mu\nu}$ around the Minkowski background at the quadratic/cubic order respectively, \mathcal{L}^{PL} stands for the so-called “pseudo-linear derivative term” (anti-symmetrization is unnormalized)

$$\mathcal{L}^{\text{PL}} \simeq \eta_{\nu_1}^{[\mu_1} \eta_{\nu_2}^{\mu_2} \eta_{\nu_3}^{\mu_3} \eta_{\nu_4}^{\mu_4]} h_{\mu_1}^{\nu_1} \partial^{\nu_2} h_{\mu_3}^{\nu_3} \partial_{\mu_2} h_{\mu_4}^{\nu_4}, \quad (3.15)$$

which was identified in [20] (see also [18, 19]).

Note in the case of no pseudo-linear derivative terms, i.e. $c_1 = 2c_5$, (3.14) seems different from the standard expansion of GR if $c_5 - c_1 \neq b_1/2$. However, this apparent discrepancy can be trivially removed by a field rescaling

$$h_{\mu\nu} = \frac{b_1}{2(c_5 - c_1)} \tilde{h}_{\mu\nu}, \quad (3.16)$$

when $c_1 \neq c_5$. The derivative terms can be recast in terms of $\tilde{h}_{\mu\nu}$ as

$$\mathcal{L}_2^{\text{der}} + \mathcal{L}_3^{\text{der}} = -\frac{b_1^3}{(c_5 - c_1)^2} \left(\mathcal{L}_2^{\text{GR}}[\tilde{h}] + \mathcal{L}_3^{\text{GR}}[\tilde{h}] + \frac{c_1 - 2c_5}{8(c_5 - c_1)} \mathcal{L}^{\text{PL}}[\tilde{h}] \right). \quad (3.17)$$

With (3.17), it becomes transparent that, up to the cubic order, the allowed derivative terms are simply the linear combination of the standard GR terms and the pseudo-linear derivative term. GR is recovered (perturbatively) for $c_1 = 2c_5$, while the pseudo-linear derivative term is recovered for $c_1 = c_5$ (where the Lagrangian is given through (3.14)).

⁴Generally, as the case of (2.36), we may write $\tilde{\alpha} \propto \alpha + \left(\frac{1}{2} + \frac{c_1 - c_5}{b_1} \right) \alpha^2 + \lambda \beta_k^2$ with arbitrary numerical constant λ such that $\tilde{\alpha}$ enters the Hamiltonian linearly. In (3.13), we fix $\lambda = -\frac{1}{2} \left(1 + \frac{3c_1 - 2c_5}{2b_1} \right)$, since with this choice of $\tilde{\alpha}$, β_i explicitly appears linearly in the Hamiltonian when $c_1 = 2c_5$ (while with other values of λ , this degeneration is not transparent), which corresponds to the GR case.

⁵More explicitly, in terms of (3.1), $\mathcal{L}_3^{\text{GR}}$ corresponds to the choice of parameters with (3.3) and $c_1 = 2c_5 = 1/4$, while \mathcal{L}^{PL} corresponds to (3.3) and $c_1 = c_5 = 1$.

As we have discussed in Sec.2.2, the diffeomorphism-breaking derivative terms must be accompanied with the appropriate potential terms. Comparing (3.13) with (2.36), in order to make these two redefinitions for $\tilde{\alpha}$ consistent, we must have

$$\frac{3c'_1 + c'_2}{b'_1} = \frac{c_1 - c_5}{b_1}, \quad (3.18)$$

which yield a constraint among c'_1, c'_2 which are parameters for the potential terms, and c_1, c_5 which are parameters for the derivative terms. Note (3.18) only involves coefficients in front of α^2 terms in (3.13) and (2.36), since β_i^2 terms in both case can be tuned freely (although we have fixed the coefficient of β_i^2 term in (3.13)) and thus be matched. Moreover, we have the same redefinition for $\tilde{\beta}_i$ (2.35) in order to make $\tilde{\alpha}$ appears linearly in the Hamiltonian coming from the potential terms. Finally, after some manipulations, the potential terms can be written as

$$\mathcal{L}_2^{\text{pot}} + \mathcal{L}_3^{\text{pot}} = 4b'_1 \left[\mathcal{L}_2^{\text{dRGT},1} + 2\frac{c_5 - c_1}{b_1} \mathcal{L}_3^{\text{dRGT},1} + 2 \left(\frac{c'_1}{b'_1} + \frac{c_1 - c_5}{b_1} \right) \mathcal{L}_3^{\text{dRGT},2} \right], \quad (3.19)$$

where $\mathcal{L}^{\text{dRGT},1}$ and $\mathcal{L}^{\text{dRGT},2}$ are the dRGT nonlinear mass terms given in (2.39) and (2.40) respectively, and again the subscripts denote the expansions at the corresponding orders. Similar to the derivative terms, for $c_5 \neq c_1$, (3.19) can be recast in terms of the same rescaled $\tilde{h}_{\mu\nu}$ as (3.16),

$$\mathcal{L}_2^{\text{pot}} + \mathcal{L}_3^{\text{pot}} = \frac{b'_1 b_1^2}{(c_5 - c_1)^2} \left[\mathcal{L}_2^{\text{dRGT},1}[\tilde{h}] + \mathcal{L}_3^{\text{dRGT},1}[\tilde{h}] + \left(1 + \frac{b_1 c'_1}{b'_1 (c_5 - c_1)} \right) \mathcal{L}_3^{\text{dRGT},2}[\tilde{h}] \right], \quad (3.20)$$

which is explicitly the linear combination of the expansions of $\mathcal{L}_3^{\text{dRGT},1}$ and $\mathcal{L}_3^{\text{dRGT},2}$.

At this point, we can understand how GR terms and the “pseudo-linear derivative term” (3.15) arise with different choices of parameters:

- GR terms

According to (3.14) or equivalently (3.17), the derivative terms reduce to the GR form when $c_1 = 2c_5$. In this case, (3.13) becomes

$$\tilde{\alpha} = \alpha + \frac{1}{2} \left(1 + \frac{c_1}{b_1} \right) (\alpha^2 - \beta_i^2), \quad (3.21)$$

which is nothing but the perturbation of lapse $\delta\tilde{N}$ corresponding to the rescaled $\tilde{h}_{\mu\nu}$ in (3.16)⁶. This is consistent with the well-known conclusion that in GR it is the lapse function that generates the Hamiltonian constraint. When there is no potential term, with the set of $\{\tilde{\alpha}, \beta_i\}$, β_i also appears linearly in the Hamiltonian (since the first two lines in (3.10) vanish) and thus generates three momentum constraints. When the potential terms are included, as has been discussed in Sec.2.2, or more generally as in (3.20), only potential terms of the dRGT form are allowed.

⁶This can be verified directly by evaluating $\delta\tilde{N} = -\frac{1}{2}\tilde{h}_{00} - \frac{1}{8}(\tilde{h}_{00}^2 - 4\tilde{h}_{0i}\tilde{h}_{0i}) + \mathcal{O}(\tilde{h}^3)$.

- Pseudo-linear derivatives

According to (3.14), this corresponds to $c_1 = c_5 \neq 0$ and $c'_2 = -3c'_1 \neq 0$, and thus both sides of (3.18) identically vanish. In this case, the cubic derivative terms are fixed to be \mathcal{L}^{PL} , the cubic potential terms (2.28) are fixed to be $c'_1 (h^3 - 3h h_{\mu\nu} h^{\mu\nu} + 2h^\mu_\nu h^\nu_\rho h^\rho_\mu) \equiv 8c'_1 \mathcal{L}_3^{\text{dRGT},2}$, which is the generalization of quadratic Fierz-Pauli mass term to the cubic order, i.e. the cubic “pseudo-linear” potential term. In this case, (3.13) reduces to

$$\tilde{\alpha} = \alpha + \frac{1}{2}\alpha^2 - \frac{1}{2} \left(1 + \frac{c_1}{2b_1}\right) \beta_i^2 = -\frac{1}{2}h_{00} - \frac{c_1}{4b_1}h_{0i}^2. \quad (3.22)$$

Since now \mathcal{H}_{3,β^2h} does not vanish, $\beta_i \equiv h_{0i}$ has already appeared quadratically in the Hamiltonian, it is essentially h_{00} that acts as the Lagrange multiplier⁷.

4 Conclusion

In this note we studied the Lorentz invariant derivative interactions for a spin-2 field, up to the cubic order. Through a perturbative ADM analysis, we determine the parameters by requiring the existence of a combination of perturbation of lapse $\alpha \equiv \delta N$ and shift β_i , which enters the Hamiltonian linearly and thus generates a constraint that may eliminate the ghost degree of freedom. Simply demanding that $\{\alpha, \beta_i\}$ have vanishing conjugate momenta yields the set of cubic derivative terms (3.14) satisfying this requirement, which corresponds to the linear combination of the standard GR terms and the “pseudo-linear derivative term” (3.15) firstly identified in [20]. The resulting derivative terms possess a Lagrange multiplier given in (3.13), which is nonlinear function of $\{\alpha, \beta_i\}$ generally and is responsible to generating the constraint that removes the ghost. After fixing the derivative terms, the potential terms should be included without violating (3.13), which yields an additional constraint on parameters (3.18), and the resulting set of potential terms corresponds to the linear combination of the first two dRGT mass terms.

Our results confirm the existence of “pseudo-linear derivative term” at cubic order, which is, however, according to the “no-go” theorem for such non-GR derivative terms [22], not the leading expansion of any ghost-free nonlinear terms. This prevents it from being a viable theory of massive gravity, since any coupling with matter will inevitably push the theory towards a fully nonlinear level. On the other hand, this “no-go” theorem makes not only GR but also the “pseudo-linear derivative term” itself very special, which deserves further investigations. The lack of nonlinear completion for the “pseudo-linear derivative term”, is reminiscent of the case of vector field(s). For example, a single vector field with $U(1)$ gauge symmetry does not have self-interactions on flat background [25], while (nonlinear) derivative interactions are allowed when there are multiple vector fields [26] or the gauge symmetry is abandoned [27]. Thus, there is a possibility that in the bi/multi-metric framework, or Lorentz-breaking context (e.g. [28, 29]), ghost-free non-GR derivative interactions may exist.

⁷In fact, the pseudo-linear terms in [20] are derived just based on the requirement that h_{00} appears linearly in the Hamiltonian.

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A Linearized Einstein-Hilbert

In order to illuminate the basic procedure in this work, here we briefly show how the linearized Einstein-Hilbert Lagrangian is uniquely determined by the constraint analysis. See [30] for a general discussion on the dependence of number of d.o.f on the kinetic terms.

Consider a general Lorentz invariant kinetic term for metric perturbation $h_{\mu\nu}$,

$$\mathcal{L}_2 = \Gamma_2 \partial_\alpha h_{\mu\nu} \partial_\beta h_{\rho\sigma}, \quad (\text{A.1})$$

where the “coefficient” Γ_2 a trinomial of Minkowski metric $\eta^{\mu\nu}$, which takes the following general form

$$\Gamma_2 = b_1 \eta^{\alpha\beta} \eta^{\mu\rho} \eta^{\nu\sigma} + b_2 \eta^{\alpha\rho} \eta^{\beta\sigma} \eta^{\mu\nu} + b_3 \eta^{\alpha\nu} \eta^{\beta\sigma} \eta^{\mu\rho} + b_4 \eta^{\alpha\beta} \eta^{\mu\nu} \eta^{\rho\sigma}, \quad (\text{A.2})$$

with four undetermined numerical coefficients b_1, \dots, b_4 . In terms of the perturbative ADM variables $\{\alpha, \beta_i, h_{ij}\}$, at the quadratic order and up to total derivatives, (A.1) yields

$$\mathcal{L}_2^{\text{ADM}} = \mathcal{L}_{2,\alpha^2} + \mathcal{L}_{2,\alpha\beta} + \mathcal{L}_{2,\beta^2} + \mathcal{L}_{2,\alpha h} + \mathcal{L}_{2,\beta h} + \mathcal{L}_{2,h^2}, \quad (\text{A.3})$$

with

$$\mathcal{L}_{2,\alpha^2} = -4(b_1 + b_2 + b_3 + b_4) \dot{\alpha}^2 + 4(b_1 + b_4) \partial_i \alpha \partial_i \alpha, \quad (\text{A.4})$$

$$\mathcal{L}_{2,\alpha\beta} = -4(b_2 + b_3) \dot{\alpha} \partial_i \beta_i, \quad (\text{A.5})$$

$$\mathcal{L}_{2,\beta^2} = (2b_1 + b_3) \dot{\beta}_i \dot{\beta}_i - b_3 \partial_i \beta_i \partial_j \beta_j - 2b_1 \partial_j \beta_i \partial_j \beta_i, \quad (\text{A.6})$$

$$\mathcal{L}_{2,\alpha h} = -2(b_2 + 2b_4) \dot{\alpha} \dot{h}_{ii} + 2\partial_j \alpha (b_2 \partial_i h_{ij} + 2b_4 \partial_j h_{ii}), \quad (\text{A.7})$$

$$\mathcal{L}_{2,\beta h} = -2b_2 \partial_i \beta_i \dot{h}_{jj} - 2b_3 \partial_j \beta_i \dot{h}_{ij}, \quad (\text{A.8})$$

$$\begin{aligned} \mathcal{L}_{2,h^2} = & -b_1 \dot{h}_{ij} \dot{h}_{ij} - b_4 \dot{h}_{ii} \dot{h}_{jj} \\ & + b_1 \partial_k h_{ij} \partial_k h_{ij} + b_2 \partial_j h_{ii} \partial_k h_{kj} + b_3 \partial_j h_{ij} \partial_k h_{ik} + b_4 \partial_j h_{ii} \partial_j h_{kk}. \end{aligned} \quad (\text{A.9})$$

In order to prevent α and β_i from being dynamical, terms in the above proportional to $\dot{\alpha}^2$, $\dot{\alpha} \partial_i \beta_i$, $\dot{\beta}_i \dot{\beta}_i$ and $\dot{\alpha} \dot{h}_{ii}$ must be identically vanishing, which implies

$$b_1 + b_2 + b_3 + b_4 = 2b_1 + b_3 = b_2 + b_3 = b_2 + 2b_4 = 0,$$

which *uniquely* fixes the four coefficients up to an overall factor

$$b_2 = 2b_1, \quad b_3 = -2b_1, \quad b_4 = -b_1. \quad (\text{A.10})$$

It is interesting to see that with these coefficients, $\partial_i \alpha \partial_i \alpha$ also drops out from the Lagrangian, which makes it as a Lagrange multiplier. The final quadratic Lagrangian is thus given by (2.20), which is just the linearization of Einstein-Hilbert term.

B Explicit expressions

Here we collect some explicit expressions, whose quantitative forms are needed in our discussion.

B.1 $\mathcal{L}_3^{\text{der,ADM}}$

Up to total derivatives, we have

$$\mathcal{L}_{3,\alpha^3} = -8 \left(\sum_{i=1}^{14} c_i \right) \alpha \dot{\alpha}^2 + 8(c_4 + c_5 + c_{12} + c_{14}) \alpha \partial_i \alpha \partial_i \alpha, \quad (\text{B.1})$$

$$\begin{aligned} \mathcal{L}_{3,\alpha^2\beta} &= -4(c_3 + 2c_6 + c_7 + 2c_8 + 2c_9 + 2c_{10} + 2c_{11} + 2c_{13}) \alpha \dot{\alpha} \partial_i \beta_i \\ &\quad -4(2c_1 + 2c_2 + c_3 + c_7) \dot{\alpha} \partial_i \alpha \beta_i. \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} \mathcal{L}_{3,\alpha\beta^2} &= 2(2c_1 + c_7 + c_8 + c_{12} + c_{13} + 2c_{14}) \alpha \dot{\beta}_i \dot{\beta}_i - 2(c_3 + c_6 + c_8) \alpha \partial_i \beta_i \partial_j \beta_j \\ &\quad + 2(c_3 + 2c_4 + 2c_6 + c_7 + 2c_9 + 2c_{11} + 2c_{12}) \dot{\alpha} \beta_i \dot{\beta}_i + 2(c_3 - c_{11} - c_{13}) \alpha \partial_i \beta_j \partial_j \beta_i \\ &\quad - 2(c_{12} + 2c_{14}) \alpha \partial_i \beta_j \partial_i \beta_j - 2(c_3 + c_7) \partial_i \alpha \beta_i \partial_j \beta_j - 4(c_4 + c_{12}) \partial_i \alpha \beta_j \partial_i \beta_j, \end{aligned} \quad (\text{B.3})$$

$$\mathcal{L}_{3,\beta^3} = (2c_6 + c_7 + 4c_9 + 2c_{11}) \beta_i \dot{\beta}_i \partial_j \beta_j + (4c_1 + c_7) \beta_i \dot{\beta}_j \partial_i \beta_j. \quad (\text{B.4})$$

$$\begin{aligned} \mathcal{L}_{3,\alpha^2h} &= -4(c_5 + c_8 + c_{10} + c_{13} + c_{14}) \dot{\alpha}^2 h_{ii} - 4(2c_2 + c_3 + c_4 + 2c_5 + c_{10}) \alpha \dot{\alpha} \dot{h}_{ii} \\ &\quad + 4(c_5 + c_{14}) \partial_i \alpha \partial_i \alpha h_{jj} + 4(c_1 + c_2) \partial_i \alpha \partial_j \alpha h_{ij} \\ &\quad + 2(b_1 - c_4 - 2c_5) \alpha^2 \partial^2 h_{ii} - 2(b_1 + c_9 + c_{10}) \alpha^2 \partial_i \partial_j h_{ij}. \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \mathcal{L}_{3,\alpha\beta h} &= -4(c_8 + c_{10} + c_{13}) \dot{\alpha} \partial_i \beta_i h_{jj} - 2(c_3 + c_7) \dot{\alpha} \partial_i \beta_j h_{ij} - 2(c_3 + 2c_6 + 2c_{11}) \dot{\alpha} \beta_i \partial_j h_{ij} \\ &\quad - 2(c_3 + c_7 + 2c_8 + 2c_{11}) \alpha \dot{\beta}_i \partial_j h_{ij} - 2(4c_2 + c_3) \dot{\alpha} \beta_i \partial_i h_{jj} - 4(c_2 + c_3 - c_{13}) \alpha \dot{\beta}_i \partial_i h_{jj} \\ &\quad + 2(2c_2 + c_3 - 2c_{10} - 2c_{13}) \alpha \partial_i \beta_i \dot{h}_{jj} + 2(c_3 + 2c_{11} - 2c_{13}) \alpha \partial_i \beta_j \dot{h}_{ij}. \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \mathcal{L}_{3,\beta^2h} &= (c_6 + c_{11} + c_{12}) \dot{\beta}_i \dot{\beta}_j h_{ij} + (c_8 + c_{13} + 2c_{14}) \dot{\beta}_i \dot{\beta}_i h_{jj} + (c_7 + 2c_{12}) \beta_i \dot{\beta}_j \dot{h}_{ij} \\ &\quad + (c_3 + 2c_4) \beta_i \dot{\beta}_i \dot{h}_{jj} - c_8 \partial_i \beta_i \partial_j \beta_j h_{kk} - c_{13} \partial_i \beta_j \partial_j \beta_i h_{kk} - 2c_{14} \partial_i \beta_j \partial_i \beta_j h_{kk} \\ &\quad - 2c_1 \partial_i \beta_j \partial_k \beta_j h_{ik} - c_7 \partial_i \beta_j \partial_k \beta_k h_{ij} - c_{12} \partial_i \beta_j \partial_i \beta_k h_{jk} - c_3 \beta_i \partial_i \beta_j \partial_j h_{kk} \\ &\quad - c_7 \beta_i \partial_i \beta_j \partial_k h_{jk} - 2c_{11} \beta_i \partial_k \beta_j \partial_j h_{ik} + 2(2b_1 - c_4) \beta_i \partial_j \beta_i \partial_j h_{kk} - 2c_6 \beta_i \partial_j \beta_j \partial_k h_{ik} \\ &\quad - 2(2b_1 + c_9) \beta_i \partial_j \beta_i \partial_k h_{jk} - 2c_{12} \beta_i \partial_j \beta_k \partial_j h_{ik}. \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \mathcal{L}_{3,\alpha h^2} &= -2(2c_5 + c_{10}) \dot{\alpha} h_{ii} \dot{h}_{jj} - 2(c_4 + c_9) \dot{\alpha} h_{ij} \dot{h}_{ij} \\ &\quad + \alpha \left\{ -2(c_1 + c_{14}) \dot{h}_{ij} \dot{h}_{ij} - 2(c_2 + c_5) \dot{h}_{ii} \dot{h}_{jj} - 4c_5 \partial^2 h_{jj} h_{kk} - 2c_4 \partial^2 h_{jk} h_{jk} \right. \\ &\quad - 2c_{10} \partial_i \partial_j h_{ij} h_{kk} - 4c_2 \partial_i \partial_j h_{kk} h_{ij} - 2c_3 \partial_i \partial_j h_{ik} h_{jk} - 2c_5 \partial_i h_{jj} \partial_i h_{kk} - 4c_2 \partial_j h_{ij} \partial_i h_{kk} \\ &\quad \left. - 2(c_4 - c_{14}) \partial_i h_{jk} \partial_i h_{jk} - 2(c_3 - c_{13}) \partial_k h_{ij} \partial_i h_{jk} + 2c_8 \partial_i h_{ij} \partial_k h_{jk} \right\}. \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned}
\mathcal{L}_{3,\beta h^2} = & 2(c_6 - c_7) \beta_i \dot{h}_{ij} \partial_k h_{jk} + (c_7 - 2c_6) \beta_i \partial_k h_{ij} \dot{h}_{jk} - c_3 \partial_j \beta_i h_{ij} \dot{h}_{kk} + 2c_8 \beta_i \partial_k h_{ik} \dot{h}_{jj} \\
& - (c_3 + 2c_8) \beta_i \dot{h}_{ik} \partial_k h_{jj} - 2(c_6 + c_{11}) \partial_k \beta_i h_{ij} \dot{h}_{jk} - 2c_{10} \partial_i \beta_j \dot{h}_{jj} h_{kk} \\
& - 2(c_8 + c_{13}) \partial_k \beta_j h_{ii} \dot{h}_{jk} - 2c_9 \partial_k \beta_k h_{ij} \dot{h}_{ij} - c_7 \partial_k \beta_i \dot{h}_{ij} h_{jk} \\
& - 2c_2 \beta_i \partial_i h_{jj} \dot{h}_{kk} - 2c_1 \beta_i \partial_i h_{jk} \dot{h}_{jk}.
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
\mathcal{L}_{3,h^3} \equiv & -c_4 h_{ij} \dot{h}_{ij} \dot{h}_{kk} - c_5 h_{ii} \dot{h}_{jj} \dot{h}_{kk} - c_{12} h_{ij} \dot{h}_{ik} \dot{h}_{jk} - c_{14} h_{ii} \dot{h}_{jk} \dot{h}_{jk} + c_1 h_{ij} \partial_i h_{kl} \partial_j h_{kl} \\
& + c_2 h_{ij} \partial_i h_{kk} \partial_j h_{ll} + c_3 h_{ij} \partial_j h_{ik} \partial_k h_{ll} + c_4 h_{ij} \partial_k h_{ll} \partial_k h_{ij} + c_5 h_{ii} \partial_k h_{ll} \partial_k h_{jj} \\
& + c_6 h_{ij} \partial_k h_{ik} \partial_l h_{jl} + c_7 h_{ij} \partial_j h_{ik} \partial_l h_{kl} + c_8 h_{ii} \partial_j h_{jk} \partial_l h_{kl} + c_9 h_{ij} \partial_k h_{ij} \partial_l h_{kl} \\
& + c_{10} h_{ii} \partial_k h_{jj} \partial_l h_{kl} + c_{11} h_{ij} \partial_k h_{jl} \partial_l h_{ik} + c_{12} h_{ij} \partial_l h_{jk} \partial_l h_{ik} \\
& + c_{13} h_{ii} \partial_k h_{jl} \partial_l h_{jk} + c_{14} h_{ii} \partial_l h_{jk} \partial_l h_{jk}.
\end{aligned} \tag{B.10}$$

We make many integration-by-parts so that (B.1)-(B.10) have already been put into the first order form (in time derivatives). The vanishing of time derivatives of α and β_i yields 12 linearly independent algebraic equations:

$$\sum_{i=1}^{14} c_i = 0, \tag{B.11}$$

$$c_3 + 2c_6 + c_7 + 2c_8 + 2c_9 + 2c_{10} + 2c_{11} + 2c_{13} = 0, \tag{B.12}$$

$$2c_1 + 2c_2 + c_3 + c_7 = 0, \tag{B.13}$$

$$2c_1 + c_7 + c_8 + c_{12} + c_{13} + 2c_{14} = 0, \tag{B.14}$$

$$2c_6 + c_7 + 4c_9 + 2c_{11} = 0, \tag{B.15}$$

$$4c_1 + c_7 = 0, \tag{B.16}$$

$$2c_2 + c_3 + c_4 + 2c_5 + c_{10} = 0, \tag{B.17}$$

$$c_8 + c_{10} + c_{13} = 0, \tag{B.18}$$

$$c_3 + c_7 + 2c_8 + 2c_{11} = 0, \tag{B.19}$$

$$4c_2 + c_3 = 0, \tag{B.20}$$

$$c_2 + c_3 - c_{13} = 0, \tag{B.21}$$

$$c_6 + c_{11} + c_{12} = 0, \tag{B.22}$$

of which the solutions are given in (3.3).

B.2 $\mathcal{F}_{ij}^{(2)}$ and $\mathcal{W}^{(3)}$

$$\begin{aligned}
\mathcal{F}_{ij}^{(2)} = & (-2c_1 \partial_k \beta_l h_{kl} - (3c_1 - 2c_5) \beta_k \partial_l h_{kl} + 2c_5 \partial_l \beta_l h_{kk} + c_1 \beta_l \partial_l h_{kk}) \delta_{ij} \\
& - (c_1 - 2c_5) \beta_i \partial_k h_{jk} + (3c_1 - 2c_5) \beta_k \partial_i h_{kj} + (c_1 - 2c_5) \beta_i \partial_j h_{kk} \\
& + 2c_1 \partial_i \beta_k h_{kj} - 2c_5 \partial_i \beta_j h_{kk} + 2c_1 \partial_k \beta_i h_{jk} - 2c_1 \partial_k \beta_k h_{ij} - c_1 \beta_k \partial_k h_{ij} \\
& + 4(c_1 - c_5) \alpha (\partial_i \beta_j - \partial_k \beta_k \delta_{ij}) + \{i \leftrightarrow j\}.
\end{aligned} \tag{B.23}$$

and

$$\mathcal{W}^{(3)} = \mathcal{W}_{\alpha\beta^2}^{(3)} + \mathcal{W}_{\alpha^2h}^{(3)} + \mathcal{W}_{\alpha h^2}^{(3)} + \mathcal{W}_{\beta^2h}^{(3)} + \mathcal{W}_{h^3}^{(3)}, \quad (\text{B.24})$$

with

$$\mathcal{W}_{\alpha\beta^2}^{(3)} = 4(c_1 - c_5) \alpha (2\partial_i \beta_i \partial_j \beta_j - \partial_i \beta_j \partial_j \beta_i - \partial_i \beta_j \partial_i \beta_j), \quad (\text{B.25})$$

$$\mathcal{W}_{\alpha^2h}^{(3)} = -2(b_1 + 2(c_1 - c_5)) \alpha^2 (\partial_i \partial_j h_{ij} - \partial^2 h_{ii}), \quad (\text{B.26})$$

$$\begin{aligned} \mathcal{W}_{\alpha h^2}^{(3)} = & \alpha \left(-4c_5 \partial^2 h_{jj} h_{kk} + 4c_1 \partial^2 h_{jk} h_{jk} + 4c_5 \partial_i \partial_j h_{ij} h_{kk} + 4c_1 \partial_i \partial_j h_{kk} h_{ij} \right. \\ & - 8c_1 \partial_i \partial_j h_{ik} h_{jk} - 2c_5 \partial_i h_{jj} \partial_i h_{kk} + 2(2c_1 - c_5) \partial_i h_{jk} \partial_i h_{jk} \\ & \left. + 4c_1 \partial_j h_{ij} \partial_i h_{kk} - 2c_1 \partial_k h_{ij} \partial_i h_{jk} - 2(3c_1 - 2c_5) \partial_i h_{ij} \partial_k h_{jk} \right), \end{aligned} \quad (\text{B.27})$$

$$\begin{aligned} \mathcal{W}_{\beta^2h}^{(3)} = & (3c_1 - 2c_5) \partial_i \beta_i \partial_j \beta_j h_{kk} - 3c_1 \partial_i \beta_j \partial_j \beta_i h_{kk} + 2c_5 \partial_i \beta_j \partial_i \beta_j h_{kk} - 2c_1 \partial_i \beta_j \partial_k \beta_j h_{ik} \\ & + 4c_1 \partial_i \beta_j \partial_k \beta_k h_{ij} - 2c_1 \partial_i \beta_j \partial_i \beta_k h_{jk} - 4c_1 \beta_i \partial_i \beta_j \partial_j h_{kk} + 4c_1 \beta_i \partial_i \beta_j \partial_k h_{jk} \\ & - 2(3c_1 - 2c_5) \beta_i \partial_k \beta_j \partial_j h_{ik} + 4(b_1 + c_1) \beta_i \partial_j \beta_i \partial_j h_{kk} + 2(5c_1 - 2c_5) \beta_i \partial_j \beta_j \partial_k h_{ik} \\ & - 4(b_1 + c_1) \beta_i \partial_j \beta_i \partial_k h_{jk} - 4c_1 \beta_i \partial_j \beta_k \partial_j h_{ik}, \end{aligned} \quad (\text{B.28})$$

$$\begin{aligned} \mathcal{W}_{h^3}^{(3)} = & c_1 h_{ij} \partial_i h_{kl} \partial_j h_{kl} - c_1 h_{ij} \partial_i h_{kk} \partial_j h_{ll} + 4c_1 h_{ij} \partial_j h_{ik} \partial_k h_{ll} - 2c_1 h_{ij} \partial_k h_{ll} \partial_k h_{ij} \\ & + c_5 h_{ii} \partial_k h_{ll} \partial_k h_{jj} - (5c_1 - 2c_5) h_{ij} \partial_k h_{ik} \partial_l h_{jl} - 4c_1 h_{ij} \partial_j h_{ik} \partial_l h_{kl} \\ & - (3c_1 - 2c_5) h_{ii} \partial_j h_{jk} \partial_l h_{kl} + 2c_1 h_{ij} \partial_k h_{ij} \partial_l h_{kl} - 2c_5 h_{ii} \partial_k h_{jj} \partial_l h_{kl} \\ & + (3c_1 - 2c_5) h_{ij} \partial_k h_{jl} \partial_l h_{ik} + 2c_1 h_{ij} \partial_l h_{jk} \partial_l h_{ik} \\ & + 3c_1 h_{ii} \partial_k h_{jl} \partial_l h_{jk} - c_5 h_{ii} \partial_l h_{jk} \partial_l h_{jk}. \end{aligned} \quad (\text{B.29})$$

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